PTC_ME_FEB_2.4_us_mp.mcdx

Finite Element Beginnings *David A. Pintur* © 2011 Parametric Technology Corp.
2 The discrete approach

Section 2.4: Assembling the Elements

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Array origin:

$$\mathrm{ORIGIN} := 1$$

The Assembled Stiffness Matrix

For the discrete spring system in the previous section, the assembled equations were created by setting the sum of all the forces at each node equal to zero and using compatibility of displacements at each node.

$$(1) \qquad k := \begin{bmatrix} 100 \\ 140 \\ 150 \\ 80 \end{bmatrix} \qquad K := \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & (k_1 + k_2 + k_3) & (-k_2 - k_3) & 0 \\ 0 & (-k_2 - k_3) & (k_2 + k_3 + k_4) & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

The matrix **K** is formed from each 2 x 2 element stiffness matrix, and relies upon the topology table which maps the element node numbering scheme to the global node numbers.

Further inspection of **K**, reveals that the stiffness matrix is formed by **additions** from each element stiffness matrix. Matrix addition, however, requires that each matrix be of the same size. This requirement is met by expanding each element stiffness matrix into the size of the global stiffness matrix through the use of the topology table, as demonstrated for the spring system below:

Element Equations

① $1 \rightarrow 1$
$2 \rightarrow 2$

$$\begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix} \cdot \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}^{<1>}$$

Element Equations in Global Form

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \end{bmatrix}^{\langle 1 \rangle}$$

② 1 → 2
2 → 3

$$\begin{pmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \cdot \begin{pmatrix} \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} F_2 \\ F_3 \end{pmatrix}^{<2>}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \\ F_3 \\ 0 \end{bmatrix}^{\langle 2 \rangle}$$

③ 1 → 2
2 → 3

$$\begin{pmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{pmatrix} \cdot \begin{pmatrix} \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} F_2 \\ F_3 \end{pmatrix}^{<3>}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \\ F_3 \\ 0 \end{bmatrix}^{\langle 3 \rangle}$$

④ 1 → 3
2 → 4

$$\begin{pmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{pmatrix} \cdot \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} F_3 \\ F_4 \end{pmatrix}^{<4>}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_3 \\ F_4 \end{bmatrix}^{\langle 4 \rangle}$$

Since each expanded element equation is of the same 4 x 4 size, the global stiffness matrix can be formed by the simple addition of each matrix. For a system of n elements, this is equivalent to

(2) $$K = \sum_i K^i \qquad i = 1 .. n$$

where **K(i)** represents the expanded stiffness matrix for element i. The addition of the element stiffness matrices is possible, since force equilibrium at each node is determined by the addition of the expanded element force vectors **F<i>**

(3) $$F = \sum_i F^{\langle i \rangle} = R \qquad i = 1 .. n$$

where **R** is the vector of applied nodal forces.

Using the above procedure for the spring system, both the global stiffness matrix and force vector are formed from the following matrix and vector additions:

Global Stiffness Matrix

(4) $$K := \left(\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \right) \cdot \frac{kip}{ft}$$

$$K = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 390 & -290 & 0 \\ 0 & -290 & 370 & -80 \\ 0 & 0 & -80 & 80 \end{bmatrix} \frac{kip}{ft}$$

The Global Force Vector

The global force vector, f, represents the applied loads at each node:

$$(5) \qquad f = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \end{bmatrix}^{\langle 1 \rangle} + \begin{bmatrix} 0 \\ F_2 \\ F_3 \\ 0 \end{bmatrix}^{\langle 2 \rangle} + \begin{bmatrix} 0 \\ F_2 \\ F_3 \\ 0 \end{bmatrix}^{\langle 3 \rangle} + \begin{bmatrix} 0 \\ 0 \\ F_3 \\ F_4 \end{bmatrix}^{\langle 4 \rangle}$$

The superscript labels on the above vectors denotes each element's force contribution at the nodes. This assembly procedure for the forces is not required, as each element force is calculated *after* the displacements have been solved. For the spring system, the left-hand side of (5) prior to the solution is known to be:

$$(6) \qquad f = \begin{bmatrix} R_1 \\ 0 \\ 0 \\ 22 \end{bmatrix} \cdot kip$$

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